CHAPTER 3: GRADE LEVEL CONSIDERATIONS

Implementation of the standards will be challenging, especially during the early phases, when many students will not have the necessary foundational skills to master all of the expected grade-level mathematics content. This chapter provides a discussion of the mathematical considerations that went into the selection of the individual standards and describes the major roles some of them play in a standards-based curriculum. It also indicates areas where students may have difficulties, and, when possible, it provides techniques for easing them. Finally, it points out subtleties to which particular attention must be paid.

The chapter includes the following categories for each of the earlier grades:

- Areas of emphasis—Targets key areas of learning (These are taken directly from the *Mathematics Content Standards.*)
- Key standards—Identifies ( ) some of the most important standards and tries to place them into context
- Elaboration—Provides added detail on these standards and on a number of related ones
- Grade-level accomplishments—Identifies areas of mathematics readiness and learning that are likely to present particular difficulties and concerns

The five strands in the *Mathematics Content Standards* (Number Sense; Algebra and Functions; Measurement and Geometry; Statistics, Data Analysis, and Probability; and Mathematical Reasoning) organize information about the key standards for kindergarten through grade seven. It should be noted that the strand of mathematical reasoning is different from the other four strands. This strand, which is inherently embedded in each of the other strands, is fundamental in developing the basic skills and conceptual understanding for a solid mathematical foundation. **It is**

important when looking at the standards to see the reasoning in all of them. Since this is the case, this chapter does not highlight key topics in the Mathematical Reasoning strand.

The section for grades eight through twelve in this chapter is organized by discipline, and only the basic ones—Algebra I; geometry; Algebra II; trigonometry; the precalculus course, mathematical analysis; and probability and statistics—are discussed in detail. The remaining courses are guided by other considerations, such as the *Advanced Placement (AP)* tests, and are outside the scope of this document.

The grade-level readiness information, which relates to difficult content areas in mathematics, is relevant to all teachers, students, and classrooms. This information will be particularly helpful in determining whether students need to be provided with specific intervention materials, and additional instruction to learn the grade level mathematics.

The Strands

The content of the mathematics curriculum has frequently been divided into categories called strands. Like most systems of categories, the strands in mathematics were developed to break the content into a small set of manageable and understandable categories. Since there is no universal agreement on the selection of the parts, the use of strands is somewhat artificial; and many different systems have been suggested. In addition, it is often difficult to restrict a particular mathematical concept or skill to a single strand. Nonetheless, this framework continues the practice of presenting the content of mathematics in five strands for kindergarten through grade seven.

Because the content of mathematics builds and changes from grade to grade, the content in any one strand changes considerably over the course of mathematics programs for kindergarten through grade seven. Thus the strands serve only as an

aid to organizing and thinking about the curriculum but no more than that. They describe the curriculum rather than define it. For the same reason the identification of strands does not mean that each is to be given equal weight in each year of mathematics education.

The general nature of each strand is described in the sections that follow.

Number Sense

Much of school mathematics depends on numbers, which are used to count, compute, measure, and estimate. The mathematics for this standard centers primarily on the development of number concepts; on computation with numbers (addition, subtraction, multiplication, division, finding powers and roots, and so forth); on numeration (systems for writing numbers, including base ten, fractions, negative numbers, rational numbers, percents, scientific notation, and so forth); and on estimation. At higher levels this strand includes the study of prime and composite numbers, of irrational numbers and their approximation by rationals, of real numbers, and of complex numbers.

Algebra and Functions

This strand involves two closely related subjects. Functions are rules that assign to each element in an initial set an element in a second set. For example, as early as kindergarten, children take collections of colored balls and sort them according to color, thereby assigning to each ball its color in the process. Later, students work with simple numeric functions, such as unit conversions that assign quantities of measurement; for example, 12 inches to each foot.

Functions are, therefore, one of the key areas of mathematical study. As indicated, they are encountered informally in the elementary grades and grow in prominence

and importance with the student's increasing grasp of algebra in the higher grades. Beginning with the first year of algebra, functions are encountered at every turn.

Algebra proper again starts informally. It appears initially in its proper form in the third grade as "generalized arithmetic." In later grades algebra is the vital tool needed for solving equations and inequalities and using them as mathematical models of real situations. Students solve the problems that arise by translating from natural language—by which they communicate daily—to the abstract language of algebra and, conversely, from the formal language of algebra to natural language to demonstrate clear understanding of the concepts involved.

Measurement and Geometry

Geometry is the study of space and figures in space. In school any study of space, whether practical or theoretical, is put into the geometry strand. In the early grades this strand includes the use of measuring tools, such as rulers, and recognition of basic shapes, such as triangles, circles, squares, spheres, and cubes. In the later grades the content extends to the study of area and volume and the measurement of angles. In high school, plane geometry is studied both as an introduction to the concept of mathematical proof and as a fascinating structure that has profoundly influenced civilization for over 2,000 years.

Statistics, Data Analysis, and Probability

This strand includes the definitions and calculations of various averages and the analysis of data by classification and by graphical displays, taking into account randomness and bias in sampling. This strand has important connections with Strand 2, Algebra and Functions, and Strand 1, Number Sense, in the study of permutations and combinations and of Pascal's triangle. In the elementary grades effort is largely limited to collecting data and displaying it in graphs, in addition to calculating simple

averages and performing probability experiments. This strand becomes more important in grade seven and above, when the students have gained the necessary skill with fractions and algebraic concepts in general so that statistics and their impact on daily life can be discussed with more sophistication than would have been possible earlier.

Mathematical Reasoning

Whenever a mathematical statement is justified, mathematical reasoning is involved. Mathematical reasoning in an inductive form appears in the early grades and is soon joined by deductive reasoning. Mathematical reasoning is involved in explaining arithmetic facts, in solving problems and puzzles at all levels, in understanding algorithms and formulas, and in justifying basic results in all areas of mathematics.

Mathematical reasoning, requiring careful, concise, and comprehensible proofs, is at the heart of mathematics and, indeed, is the essence of the discipline, differentiating it from others. Students must realize that assumptions are always involved in reaching conclusions, and they must recognize when assumptions are being made. Students must develop the habits of logical thinking and of recognizing and critically questioning all assumptions. In later life such reasoning skills will provide students with a foundation for making sound decisions and give them an invaluable defense against misleading claims.

CHAPTER 3: PREFACE TO KINDERGARTEN THROUGH GRADE SEVEN

Mathematics, in the kindergarten through grade seven curriculum, starts with basic material and increases in scope and content as the years progress. It is like an inverted pyramid, with the entire weight of the developing subject resting on the core provided in kindergarten through grade two, when numbers, sets, and functions are introduced. If the introduction of the subject in the early grades is flawed, then later on, students can have extreme difficulty progressing; and their mathematical development can stop prematurely, leaving them, in one way or another, unable to fully realize their potential.

Because the teaching of mathematics in the early grades is largely synonymous with the problems given to the students, it is essential that students be presented with carefully constructed and mathematically accurate problems throughout their school careers. Problems which appear correct can actually be wrong, leading to serious misunderstandings on the part of the students. For example, the teacher might present the kindergarten standard for Algebra and Functions 1.1: “Identify, sort, and classify objects by attribute and identify objects that do not belong to a particular group.” At first glance, the following exercise might seem appropriate for this standard:

A picture of three objects, a basketball, a bus, and a tennis ball, is shown to the students, and they are asked to tell which one does not belong.

This statement appears to present a perfectly reasonable problem. The difficulty is that, as stated, the question is not a problem in mathematics. From a mathematical point of view, the question is to determine which of these objects belongs to one set while the third belongs to a different one. It must be clear that unless the sets are

specified in some way, the question cannot have a reasonable answer. In this case, the student must *guess* that the teacher is asking the student to sort objects by shape. The following might be asked instead: *We want to collect balls. Which of these objects should we select?* Or perhaps the contrapositive*, Which of these objects should not be included?* Another approach is to add colors; for example, coloring the bus and tennis ball blue and the basketball brown. Then a different question might be asked*: We want blue things. Which of these objects do we want?* or *We want round, blue objects. Which of these do we want?* But a question in the mathematics part of the curriculum should not be asked when the assumptions underlying what is wanted are not clearly stated.

In another example, the standard for Statistics, Data Analysis, and Probability 1.2 asks students to identify, describe, and extend simple patterns involving shape, size, or color, such as a circle or triangle or red or blue. A possible problem illustrating the standard follows:

> *The students are given a picture that shows in succession a rectangle, triangle, square, rectangle, triangle, square, blank, triangle, square. The students are asked to fill in the blank.*

While this problem may seem to be a reasonable one (and an example of problems that all too commonly appear in the mathematics curricula of the lower grades), it cannot be considered a problem in mathematics. From a mathematical point of view, there is no correct answer to this problem unless more data are supplied to the students. Mathematics is about drawing logical conclusions from explicitly stated hypotheses. *Because there is no statement about the nature of the pattern in this*

case (e.g., does the pattern repeat itself every three terms? every seven terms? every nine terms?), students can only guess at what should be in the blank spot.

The intent of the problem was probably to ask students to infer from the given data that the pattern, in all likelihood, repeats itself every three terms, leaving students to assume that a rectangle belongs in the blank spot. But if students were to start thinking that every mathematical situation always contains a hidden agenda for them to guess correctly before they can proceed, then both the teaching and learning of mathematics would be tremendously compromised. Observations from some university-level mathematicians suggest that this outcome may have already occurred with some students. Students' reluctance to take mathematical statements at face value has become a major stumbling block.

In an attempt to make mathematics "more relevant," problems described as "real world" are often introduced. The following example of such a problem is similar to many fourth grade assessment problems: *The picture below shows a 5 $\times$ 5 section of an array of lockers with only the 3 $\times$ 3 center group numbered.*

11	12	13
20	21	22
29	30	31

Figure 1

Students are given the following assessment task: *Some of the numbers have fallen off the doors of some old lockers. Figure out the missing numbers and describe the number pattern.*

This problem does not make sense mathematically. The data given are insufficient to find a unique answer. In fact, the expected "solution," as shown in figure 2, makes use of the *hidden assumption* that the array was rectangular. However, the assumptions that are given do not indicate that this is the case, and it would be improper, mathematically, to also assume that the array is rectangular.

	1	2	3	4	5
	10	11	12	13	14
	19	20	21	22	23
	28	29	30	31	32
	37	38	39	40	41

Figure 2

There are many other solutions without this assumption. For example, one is shown in figure 3.

3613	1	2	3	4	5	6	7				8	9
3614	10	11	12	13	14	15	16				17	18
3615	19	20	21	22	23	24	25				26	27
3616	28	29	30	31	32	33	34	35	36	37	38	39
3617	40	41	42	43	44	45	46	47	48	49	50	51

3618 Figure 3

3619 One of the key points of mathematics is to promote critical thinking. Students have
3620 to learn to reason precisely with the data given so that if assumptions are hidden,
3621 they know they must seek them out and question them.

3622 These remarks are not meant to diminish the importance of learning the number
3623 system and basic arithmetic, both of which are crucial as well. Here, too, these topics
3624 present problems for the kindergarten through grade seven curriculum, but not to the
3625 same degree as in many of the other areas discussed previously.

3626 The intent of the material that follows in this chapter is to try to place into correct
3627 perspective much of the material taught in these grades, to indicate where problems
3628 might be encountered with some of the most important of these topics, and to
3629 suggest some ways of resolving the difficulties. In addition, throughout this chapter
3630 some items are pointed out to show where careful development will help foster
3631 critical thinking, and suggestions are given for accomplishing this process.

Chapter 3: Kindergarten Areas of Emphasis

By the end of kindergarten, students understand small numbers, quantities, and simple shapes in their everyday environment. They count, compare, describe, and sort objects and develop a sense of properties and patterns.

Number Sense

1.0 1.1 1.2 1.3

2.0 **2.1**

3.0 3.1

Algebra and Functions

1.0 **1.1**

Measurement and Geometry

1.0 1.1 1.2 1.3 1.4

2.0 2.1 2.2

Statistics, Data Analysis, and Probability

1.0 1.1 **1.2**

Mathematical Reasoning

1.0 1.1 1.2

2.0 2.1 2.2

Chapter 3: Kindergarten

Key Standards

NUMBER SENSE

The Number Sense standard that follows is basic in kindergarten:

1.0 Students understand the relationship between numbers and quantities (i.e., that a set of objects has the same number of objects in different situations regardless of its position or arrangement).

A key skill within this standard is to group and compare sets of concrete items and recognize whether there are more, fewer, or an equal number of items in different sets. The following Number Sense standard is also important:

2.1 Use concrete objects to determine the answers to addition and subtraction problems (for two numbers that are each less than 10).

The object of these standards is to begin to develop a precise sense of what a number is. Although students at this stage are dealing mainly with small numbers, they also need experience with larger numbers. An activity to provide this experience is to have the teacher fill glass jars with tennis balls, ping-pong balls, or jelly beans and ask the students to guess how many of these items are in the glass jar. Activities such as this one help give students an understanding of magnitude of numbers and help them gain experience with estimation.

When presenting this activity, teachers need to be aware that students can get the misconception that large numbers are only approximate rather than corresponding to exact quantities. This is a serious problem that has the potential to cause real difficulty later.

One way of avoiding this difficulty is to have the students use manipulatives, such as blocks, to compare two (relatively) large numbers; for example, 14 and 15. The

class can explore the fact that 14 breaks up into two equal groups of 7, while 15 cannot be broken into two equal groups. The students would begin to appreciate that although visually distinguishing 15 objects from 14 without careful counting is difficult, the two numbers, nonetheless, are quite different. This activity should help students develop an awareness that each whole number is unique and will help them meet Number Sense Standard 1.2, which requires them to count and represent objects up to 30.

ALGEBRA AND FUNCTIONS

The role of the Algebra and Functions standard is also basic:

1.1 Identify, sort, and classify objects by attribute and identify objects that do not belong to a particular group (e.g., all these balls are green, those are red).

Although kindergarten teachers may not think of themselves as algebra teachers, they actually begin the process. They make students aware of the existence of patterns by giving them their first experience of finding them in data, by providing their initial exposure to functions, and by introducing them to abstraction. For example, students realize that a blue rectangular block and a blue ball, which obviously have different physical attributes, can nevertheless be sorted together because of their common color. This realization is the beginning of abstract reasoning, which is a higher-order thinking skill.

STATISTICS, DATA ANALYSIS, AND PROBABILITY

This standard interacts with the following Statistics, Data Analysis, and Probability standard:

1.2 Identify, describe, and extend simple patterns (such as circles or triangles) by referring to their shapes, sizes, or colors.

Elaboration

The kindergarten teacher is likely to find that many students can learn more material than is specified in the kindergarten standards. For example, the standard for committing addition and subtraction facts to memory appears in the first grade. Because committing facts to memory requires substantial amounts of practice over an extended period, memorizing addition and subtraction facts can begin in kindergarten with simple facts, such as +1s, +2s, –1s, or sums to 10. Any practice of addition and subtraction facts should be limited to these more simple problems. Likewise, students can be taught the meaning of the symbols +, -, and = in the context of addition or subtraction, but again the focus is on small numbers. In measurement, the months can be taught in kindergarten as students learn the days of the week.

Considerations for Grade-Level Accomplishments in Kindergarten

Kindergarten is a critical time for children who, when they enter school, are behind their peers in the acquisition of skills and concepts. Efficient teaching in kindergarten can help prepare these children to work at an equal level with their peers in the later grades.

Students who enter kindergarten without some background in academic language (the language of tests and texts) and an understanding of the concepts such language represents have a great disadvantage in learning mathematics. Critical for beginning mathematical development are attributes, such as color, shape, and size; abstract concepts, such as *some, all,* and *none;* and ordinal concepts, such as *before, after, yesterday,* and *tomorrow*. Teachers need clear directions on how to maximize progress in mathematics for students with limited understanding of language concepts or for students who know the concepts in their native language but do not yet know the English words for them. Kindergarten provides many

opportunities for teachers to teach basic mathematics vocabulary and concepts during instructional time or playtime; for example, students learn to take turns during a game or line up for recess (first, second, third), count off in a line (one, two, three), or determine the number of children who can take six balls out for recess if each child gets a ball (matching sets).

The most important mathematical skills and concepts for children in kindergarten to acquire are described as follows:

- Counting**.** Before beginning instruction in counting, teachers should determine the number to which the child can already count and whether the child understands what each number represents. The teacher models the next few numbers in the sequence (e.g., 5, 6, 7); provides practice for the children in saying the counting sequence through the new numbers (1, 2, 3, 4, 5, 6, 7); and matches each number to a corresponding set of objects. After a student has mastered the sequence including the new numbers, the teacher introduces several more numbers and follows the same procedure. Even though the standard requires a mastery of counting only to 30, daily practice in counting can be provided until students can count to 50 or 100 so that they may be better prepared for the challenges of the first grade.
- Reading numerals**.** The teacher should introduce numerals after the children can count to 10. Confusion between numeral names and the counting order can be *decreased* if the teacher does not introduce the numerals in order. For example, the teacher introduces the numeral 4 and then 7. For several days the teacher introduces a new numeral until the students can identify the numerals 1 through 10. The teacher should provide cumulative practice by having students review previously introduced numbers while he or she presents a new number.

- Writing numerals**.** The standards require that students know the names of the numerals from 1 to 9 and how to write them. Generally, writing numbers will require a good deal of practice; and at this age some children may have difficulty with coordination. First, students should copy a numeral many times. Then they should write it with some prompts (e.g., dots or arrows); and later they should write it from memory, with the teacher saying the number and the student writing the numeral. A multisensory approach is very important here. Teachers must encourage the students of this age not to be concerned about the quality of their handwriting as they write numerals. Young children do not yet have fully developed fine-motor skills. Many students become frustrated by the discrepancy between what they want to produce on paper and what they can actually produce.

- Understanding place value—reading numbers in the teens**.** To read and write numbers from 10 to 20, students will need to understand something about place value. The teacher can expect the numbers 11, 12, 13, and 15 to be more troublesome than 14, 16, 17, 18, and 19. The second group is regular in pronunciation (e.g., *four*teen, *six*teen), but the first group is irregular; twelve is not pronounced as “twoteen” but as “twelve.”

  An important prerequisite for understanding place value is being able to answer fact questions verbally; for example, what is 10 + 6? When the students know the facts about numbers in the teens that are regular in pronunciation, the teacher can introduce one number with irregular pronunciation and mix it with the regular numbers in a verbal exercise. New irregular numbers can be introduced as students demonstrate knowledge of previously introduced facts about numbers in the teens. Reading and writing these numbers can be introduced when students are able to do the verbal exercises.

- Learning the days of the week**.** The days of the week can be taught in a manner similar to that for counting, in which the teacher models a part of the sequence of days (Monday, Tuesday, Wednesday); provides practice in saying the sequence; introduces a new part after several days (Thursday, Friday); provides practice with this part; and then repeats the sequence from the beginning. The months of the year can also be taught in kindergarten. Unless the students have a firm understanding of the sequence of days and months, they will have difficulty with items applying concepts of time, such as *before* and *after* as indicated in the second part of the following standard:

MEASUREMENT AND GEOMETRY

1.0 Students understand the concept of time and units to measure it; they understand that objects have properties, such as length, weight, and capacity, and that comparisons may be made by referring to those properties.

Chapter 3: Grade One Areas of Emphasis

By the end of grade one, students understand and use the concept of ones and tens in the place value number system. Students add and subtract small numbers with ease. They measure with simple units and locate objects in space. They describe data and analyze and solve simple problems.

Number Sense

1.0 **1.1** **1.2** 1.3 1.4 1.5

2.0 **2.1** **2.2** **2.3** **2.4** **2.5** 2.6 2.7

3.0 3.1

Algebra and Functions

1.0 1.1 1.2 1.3

Measurement and Geometry

1.0 1.1 1.2

2.0 2.1 2.2 2.3 2.4

Statistics, Data Analysis, and Probability

1.0 1.1 1.2

2.0 **2.1**

Mathematical Reasoning

1.0 1.1 1.2

2.0 2.1 2.2

3.0

Chapter 3: Grade One

Key Standards

NUMBER SENSE

The following Number Sense standard is basic:

1.1 Count, read, and write whole numbers to 100.

It is important that students gain a conceptual understanding of numbers and counting, not simply learn to count to 100 by rote. They need to understand, for example, that counting can occur in any order and in any direction, not just in the standard left-to-right counting pattern, as long as each item is tagged once and only once. Students must understand that numbers represent sets of specific quantities of items. Of particular importance is learning and understanding the counting sequence for numbers in the teens and multiples of ten. It should be emphasized that numbers in the teens represent a ten value and a certain number of unit values—12 does not merely represent a set of 12 items; it also represents 1 ten and 2 ones. A related and equally important Number Sense standard is:

1.2 Compare and order whole numbers to 100 by using the symbols for less than, equal to, or greater than (<, =, >).

The continuing development of addition and subtraction skills as described in the following standards is basic:

2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

2.5 Show the meaning of addition (putting together, increasing) and subtraction (taking away, comparing, finding the difference).

For example, students should understand that the equation 15 - 8 = 7 is the same as 15 = 7 + 8. Particular attention should be paid to the assessment of these

competencies because students who fail to learn these topics will have serious difficulties in the later grades. The achievement of these standards will require that students be exposed to and asked to solve simple addition and subtraction problems throughout the school year.

STATISTICS, DATA ANALYSIS, AND PROBABILITY

The following Statistics, Data Analysis, and Probability standard is also important, but it has to be handled carefully:

2.1 Describe, extend, and explain ways to get to a next element in simple repeating patterns (e.g., rhythmic, numeric, color, and shape).

Students should *never* get the idea that the next term *automatically* repeats (unless they are told explicitly that it does); however, it is legitimate to ask what is the *most likely* next term. In this way students begin to learn not only the usefulness of patterns in sorting and understanding data but also careful, precise patterns of thought. Examples are sequences of colors, such as red, blue, red, blue, . . . or numbers, 1, 2, 3, 1, 2, 3, 1, 2, 3, . . . But more complex series might also be used, such as 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, . . .

Elaboration

Teaching students to solve basic addition and subtraction problems effectively and to commit the answers to memory will require considerable practice in solving these problems. As described in Chapter 4, the associated practice should be in small doses each day or, at the very least, several times a week. At the beginning of the school year, practice should focus on smaller problems (with sums less than or equal to ten). Large-valued problems should be emphasized in practice once students are skilled at solving the easier problems. Frequent assessment should be provided to determine whether students are mastering new facts and retaining those taught

previously. Students have mastered basic facts when they can solve problems involving those facts quickly and accurately. Accurate but slow problem solving indicates that students are still using counting or other procedures to solve simple problems and have not yet committed the basic facts to memory.

Committing the basic addition and subtraction facts to memory is a major objective in the first and second grades. Students who do not commit the basic facts to memory will be at a disadvantage in further work with numbers and arithmetic.

Understanding the symmetric relationship between sets of simple addition problems, such as 7 + 2 and 2 + 7, can be used to reduce the memorization load in learning facts. The teaching of these relationships is to be incorporated into the sequence for teaching students simple addition and during their practice. For example, after students have learned 7 + 2, they can be shown that the same answer applies to 2 + 7. Moreover, by placing problems such as 7 + 2 and 2 + 7 in sequence in practice sheets, students will have the opportunity to “discover” and reinforce this relationship as well. Later, they might learn that the combination of 7, 2, and 9 can be used to create subtraction facts and addition facts.

While the standard calls for counting by 1 to 100 in the first grade, counting into the 100s can begin in the latter part of the first grade if students have mastered counting to 100. Counting backward for numbers up to 100 should also be done in the first grade once students have mastered counting forward.

Considerations for Grade-Level Accomplishments in Grade One

The most important mathematical skills and concepts for children in grade one to acquire are described as follows:

- Reading and writing of numbers. Many students demonstrate a lack of understanding of place value when they encounter numbers such as 16 and 61. If students are confused by two such similar numbers, teachers should try to

determine whether the cause of the confusion is students' failure to understand that numbers are read from left to right or students' inadequate understanding of place value. Instruction should be carefully sequenced to show that 16 is 1 ten and 6 ones, while 61 is 6 tens and 1 one. Students need to know prerequisite skills underlying place value, such as 6 tens equals 60 and its corollary, 60 equals 6 tens, and addition facts in which a single-digit number is added to the tens number, 10 + 3, 10 + 5, 30 + 6. These facts can be taught verbally before students read and write the numbers.

Learning the number that represents a group of tens is important for understanding place value and reading numbers. Some students are more likely to have difficulty with groups of tens in which the tens number does not say the name of the first digit (e.g., "twenty" is not pronounced "twoty") than with tens numbers in which the name of the first digit is pronounced, sixty, forty, seventy, eighty, ninety. Teachers should provide more practice on the more difficult items.

- Skip counting**.** In addition to enhancing children's number sense, skip counting is important for facilitating the learning of multiplication and division. Counting by tens should be introduced when students can count by ones to about 20 or 30. Counting by tens helps students learn to count by ones to 100. Skip counting is taught just like counting by ones. The teacher models the first part of the sequence; then the students practice the first part. The modeling and practicing continue on new parts of the sequence until students can say the whole sequence. Skip counting requires systematic teaching using a procedure similar to that discussed for counting by ones. Regularly scheduled practice will help students master counting a sequence. Previously introduced sequences should be reviewed as students learn new ones.

DRAFT: January 28, 2005

- Teaching of addition and subtraction facts**.** Teaching addition and subtraction facts and making assessments should be systematic, as was discussed previously.
- Understanding of symmetric relationships**.** Understanding the symmetric relationship of facts can reduce the number of facts to be memorized in learning.
- Adding and subtracting of one- and two-digit numbers**.** Students can be helped to avoid difficulties with adding one- and two-digit numbers if they are given practice with “lining up” numbers in the problem and adding from right to left. This procedure can be confusing to students because (as previously discussed) we read and write numbers from left to right. Furthermore, in anticipation of subtracting one- and two-digit numbers, students need practice in working from top to bottom.

Chapter 3: Grade Two Areas of Emphasis

By the end of grade two, students understand place value and number relationships in addition and subtraction, and they use simple concepts of multiplication. They measure quantities with appropriate units. They classify shapes and see relationships among them by paying attention to their geometric attributes. They collect and analyze data and verify the answers.

Number Sense

1.0 **1.1** 1.2 **1.3**

2.0 **2.1** **2.2** 2.3

3.0 **3.1** **3.2** **3.3**

4.0 **4.1** **4.2** **4.3**

5.0 **5.1** **5.2**

6.0 6.1

Algebra and Functions

1.0 **1.1** 1.2 1.3

Measurement and Geometry

1.0 1.1 1.2 **1.3** 1.4 1.5

2.0 **2.1** **2.2**

Statistics, Data Analysis, and Probability

1.0 1.1 1.2 1.3 1.4

2.0 2.1 2.2

Mathematical Reasoning

1.0 1.1 1.2

DRAFT: January 28, 2005

2.0 2.1 2.2

3.0

Chapter 3: Grade Two

Key Standards

NUMBER SENSE

As was the case in grade one, the students' growing mastery of whole numbers is the basic topic in grade two, although fractions and decimals now appear. These Number Sense standards are particularly important:

1.1 Count, read, and write whole numbers to 1,000 and identify the place value for each digit.

1.3 Order and compare whole numbers to 1,000 by using the symbols <, =, >.

For many of the same reasons, the standards listed below are very important:

2.1 Understand and use the inverse relationship between addition and subtraction (e.g., an opposite number sentence for 8 + 6 = 14 is 14 – 6 = 8) to solve problems and check solutions.

2.2 Find the sum or difference of two whole numbers up to three digits long.

Standard 2.1 gives students a clear application of the relations between different types of operations (addition and subtraction) and can be used to encourage more flexible methods of thinking about and solving problems; for example, a knowledge of addition can facilitate the solving of subtraction problems and vice versa. The problem 144 - 98 = ? can be solved by realizing that 144 = 100 + 44 = 98 + 2 + 44 = 98 + 46.

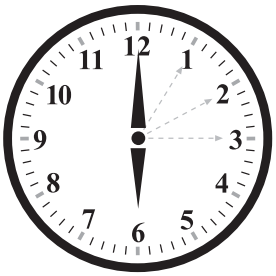
Standard 2.2 asks for the teaching of the addition algorithm for numbers up to three digits. For children at this age, two things should be observed. One is that the teaching should be flexible at the beginning and should not insist on the formalism of that algorithm from the start. For example, one can begin the teaching of 23 + 45 by considering 20 + 3 + 40 + 5 = 20 + 40 + 3 + 5 = 60 + 8 = 68. This process gets

children used to the advantage of adding the tens digits and the ones digits separately. A second thing is not to emphasize the special skill of "carrying" at the initial stage. The key idea of this algorithm is the ability to add the numbers column by column, one digit at a time. In other words the important thing is to be able to add digits of the same place (ones digits, tens digits, hundreds digits, etc.) and still obtain the correct answer at the end. Only after this idea has sunk in should the "carrying" skill be taught. The same remark applies to the subtraction algorithm: teachers should emphasize at the beginning the fact that the subtraction of two three-digit numbers can be obtained by performing single-digit subtractions. Thus, 746 – 503 can be computed from three single-digit subtractions: 7 – 5 = 2, 4 – 0 = 4, and 6 – 3 = 3 so that 746 – 503 = 243. Show that this is because 746 – 503 = 700 + 40 + 6 – 500 – 00 – 3. The special skill of "trading" needed for a subtraction of 793 – 568 can be taught only after the idea of the efficacy of single-digit subtractions has taken root. Formal explanations at this grade level are not necessary; friendly persuasion is the order of the day. The mathematical reasoning behind these algorithms is taken up in grade four.

The third Number Sense standard is basic to students' understanding of arithmetic and the ability to solve multiplication and division problems:

3.0 Students model and solve simple problems involving multiplication and division.

Here, fluency with skip counting is helpful. It is important to remind students that multiplication is a shorthand for repeated addition: the meaning of 5 x 7 is exactly 7 + 7 + 7 + 7 + 7, no more and no less. This is an opportunity to impress on students the fact that every symbol and every concept in mathematics have a precise, unambiguous meaning.



The discussion of fractions and the goals represented in Number Sense Standards 4.1, 4.2, and 4.3 are also essential features of students' developing arithmetical competencies. Although equivalence of fractions is not explicitly presented in the standards, it is also a good idea to begin the discussion of the topic at this point—students should know, for example, that $^2/_4$ is the same as $^1/_2$, a concept that can (and should) be demonstrated with pictures. Finally, as a practical matter and as a basic application of the topics discussed previously, the material in Number Sense Standards 5.1 and 5.2—on modeling and solving problems involving money—is very important. Borrowing money gives a practical context to the concept of subtraction. Special attention should be paid to the need for introducing the symbols $ and ¢ and to the fact that the order of the symbol for dollars is $3, not 3$; but for cents, the order is 31¢, not ¢31.

ALGEBRA AND FUNCTIONS

In the Algebra and Functions strand, the following standard is an *essential* feature of mathematics instruction in grade two:

1.1 Use the commutative and associative rules to simplify mental calculations and to check results.

However, the emphasis here should be on the *use* of these rules to simplify; for example, knowing that 5 + 8 = 13 saves the labor of also learning that 8 + 5 = 13. Learning the terminology is not nearly as important. The students should begin to develop an appreciation for the power of unifying rules; but overemphasizing these

topics, particularly the sophisticated concept of the associative rule, is probably worse than not mentioning them at all.

MEASUREMENT AND GEOMETRY

Although Standard 1.3 listed below from the Measurement and Geometry strand is important, more emphasis should be given to the topics in Standard 2.0.

1.3 Measure the length of an object to the nearest inch and/or centimeter.

2.0 Students identify and describe the attributes of common figures in the plane and of common objects in space.

Because understanding spatial relations will be more difficult for some students than for others (especially the concepts involving three-dimensional information), teachers should carefully assess how well students understand these shapes and figures and their relationships.

STATISTICS, DATA ANALYSIS, AND PROBABILITY

Although Standard 1.0 in the Statistics, Data Analysis, and Probability strand is important for grade two, the topics in Standard 2.0 are more important in this grade.

1.0 Students collect numerical data and record, organize, display, and interpret the data on bar graphs and other representations.

2.0 Students demonstrate an understanding of patterns and how patterns grow and describe them in general ways.

But here, as for grade one, it is important that students distinguish between the most likely next term and *the* next term. In statistics students look for likely patterns, but in mathematics students need to know the rule that generates the pattern to determine “the” next term. As an example, given only the sequence 2, 4, 6, 8, 10, students should *not* assert that the next term is 12 but, instead, that the most likely

next term is 12. For example, the series might have actually been 2, 4, 6, 8, 10, 14, 16, 18, 20, 22, 26, 28 . . . . The ability to distinguish between what is likely and what is given promotes careful, precise thought.

Elaboration

In the second grade, work on committing the answers to basic addition and subtraction problems to memory should continue for those students who have not mastered them in the first grade. Students' knowledge of facts needs to be assessed at the beginning of the school year. The assessment could be done individually so that the teacher can determine whether the student has committed the facts to memory. Mastery of addition and subtraction facts can also be assessed with simple paper-and-pencil tests. Students should be asked to solve a whole sheet of problems in one or two minutes. As noted earlier, students who have committed the basic facts to memory will quickly and correctly dispose of these simple tasks. If not, they are, most likely, solving the problem by counting in their head (Geary 1994) or using time-consuming counting procedures to generate answers. Additional practice will be necessary for these children.

Students learn the basics of how to "carry" and "borrow" in the second grade. Because carrying and borrowing are difficult for students to master, extended discussion and practice of these skills will likely be necessary (Fuson and Kwon 1992). To carry and borrow correctly, it is important that students understand the base-10 structure of the number system and the concept that carrying and borrowing involve exchanging sets of 10 ones or 10 tens and so forth from one column to the next. It is common for students to incorrectly conceptualize carrying or borrowing; for example, taking a one from the tens column and giving it to the ones column. What has been given, in fact, is one set of 10 units, not one unit from the tens. For example, borrowing in the case of 43 - 7 can be explained as follows: 43 - 7 =

(30 + 13) - 7 = 30 + (13 - 7) = 30 + 6 = 36, illustrating the associative law of addition in the process. Initially, problems should be limited to those that require carrying or borrowing across one column (e.g., 17 + 24, 43 - 7), and particular attention should be paid to problems with zero (90 - 34 and 94 - 30) because they are often confusing to students (VanLehn 1990).

Multiplication is introduced in the second grade, and students are to commit to memory the twos, fives, and tens facts. During the initial learning of multiplication, students often confuse addition and multiplication facts, but these confusions should diminish with additional practice. These facts should be taught with the same systematic approach as was discussed for the addition facts in grade one. *The skip counting series for numbers other than 2, 5, and 10 (e.g., 3s, 4s, 9s, 7s, 25s) can be introduced in the second grade to prepare students for learning more multiplication facts in the third grade.* Additionally, the associative and commutative laws can be used to increase the number of multiplication facts the students know. For example, there is no need for students to learn 5 x 8 if they already know 8 x 5.

Students in these early grades often have trouble lining numbers up for addition or subtraction. Reminding students to make sure that their numbers are lined up evenly is essential. Students can be taught to use estimation to determine whether their answers are reasonable. However, it is unwise to try to put undue emphasis on estimation by teaching second grade students to answer problems only by making estimates. Instead, they should concentrate on problems that demand an exact answer and use estimation to check whether their answer is reasonable.

The work with fractions should include examples showing fractions that are less than one, fractions that are equal to one, and fractions that are equal to more than one. This range is needed to prevent students from thinking that fractions express only units less than one. To this end, teachers need to make sure that students can

freely work with improper fractions and understand that, the name notwithstanding, there is nothing wrong with improper fractions.

It has been pointed out that many second grade students have real difficulty with the written form of fractions but much less trouble with their verbal descriptions. Therefore, the verbal descriptions should be emphasized at this level, although students will, of course, eventually need to know the standard written representations of fractions.

Considerations for Grade-Level Accomplishments in Grade Two

The most important mathematical skills and concepts for children in grade two to acquire are described as follows:

- Counting**.** Many students require careful teaching of counting from 100 through 999. Students can learn the counting skills for the entire range through exercises in which the teacher models and provides practice sets consisting of series. First, the teacher models numbers within a particular decade (e.g., 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360). A daily teaching session might include work on several series (e.g., 350 to 360, 140 to 150, 470 to 480). Sets within a decade would be worked on daily until students demonstrate the ability to generalize to new series. During the next stage students would practice on series in which they move from one decade to the next (e.g., 365 to 375, 125 to 135, 715 to 725). Students may have difficulty making the transition from one decade to the next without explicit instruction and adequate practice. When the students demonstrate a general ability to make this transition, the final set of series would be introduced. These sets would include those in which the transition from one one-hundred number to the next occurs: 595 to 605, 195 to 205, 495 to 505.
- Writing numbers**.** If the students are not instructed carefully, some may develop the misconception that the presence of two zeros creates a hundreds number.

These students will write three hundred twenty-five as 30025. Teachers should watch for this type of error and correct it immediately. Examples with and without zeros need to be modeled and practiced.

- Borrowing**.** Practice with the terms *more* and *less* and *top* and *bottom* should precede the introduction of problems involving borrowing. These concepts need to be firmly understood if students are to succeed with borrowing problems.
- Skip counting**.** Students should be given opportunities to skip count forward, backward, and starting at any number. Otherwise, students may develop misunderstandings such as it is not possible to count by 2s from an odd number. During the year, students should learn that skip counting by a number starting from zero will also provide a list of multiples for the number. In the process of using skip counting to learn multiples, students may become confused by numbers that appear on several lists. For example, when numbers are counted by threes and fours, the number 12 appears as the fourth number on the “multiples of three” list and as the third number on the “multiples of four” list. To avoid confusing their students, teachers should provide extensive practice with one of these sequences before introducing the next.
- Counting groups of coins**.** This process requires that students be able to say the respective count by series for the value of each coin and be able to answer addition fact questions easily, such as 25 + 5, 30 + 10, in which a nickel or dime is added to a number ending in 5 or 0. Exercises in counting coins should be coordinated with instruction in counting facts so that students have already practiced the skill thoroughly before having to apply it. Counting coins should be reviewed and extended to include quarters along with dimes, nickels, and pennies. A particular fact that some students find difficult to comprehend is adding ten to a two-digit number ending in 5 (e.g., 35 + 10).

- Aligning columns**.** Students may need systematic instruction in rewriting problems written as a column problem; practice in rewriting horizontal equations, such as 304 + 23 =__ or 6 + 345 = ___, in column form; and help in lining numbers up for addition or subtraction. In certain situations they can be taught to use estimation to check whether their answers are reasonable and, if not, to recheck their work to find their mistakes. As was discussed previously in the subsection on elaboration, it is unwise to try to teach students in grade two to answer problems that request only an estimate as the answer. Students need to become accustomed to obtaining exact answers and using estimation only as an aid to check whether the answer is reasonable.

- Understanding associativity. Students are expected to know and use the associative attribute of addition and multiplication in early grades. It is already discussed in the second grade Algebra and Functions, Standard 1.1 (addition) and in the third grade Algebra and Functions, Standard 1.5 (multiplication). Associativity often helps to simplify mental calculations or to verify the correctness of the results and, therefore, its usefulness in those grades.

  However, once subtraction and division are introduced, the teacher should demonstrate to the students that associativity does not hold for subtraction and division. For example, given the simple subtraction sentence 9 – 4 – 2, one cannot arbitrarily group the operands because (9 – 4) – 2 *is not* equal to 9 – (4 – 2). Similarly, in a division sentence such as 18 ÷ 2 ÷ 3, (18 ÷ 2) ÷ 3 *is not* equal to 18 ÷ (2 ÷ 3). Such demonstrations, not necessarily in-depth teaching, should occur no later than in the second grade for subtraction and in the fourth grade for division.

- Reviewing time equivalencies**.** Students will need to review time equivalencies (e.g., 1 minute equals 60 seconds, 1 hour equals 60 minutes, 1 day equals 24

hours, 1 week equals 7 days, 1 year equals 12 months). These equivalencies need to be practiced and reviewed so that all students are able to commit them to memory.

- Understanding money**.** In the teaching of decimal notation for money, teachers must ensure that students can read and write amounts such as $2.05, in which there is a zero in the tenths column, and $.65, in which there is no dollar amount. By the end of the second grade, students should be able to write ten cents as $.10 and ten dollars as $10.00 in decimal notation.

- Telling time**.** Students can be taught a general procedure for telling time. Telling time on an analog clock can begin with teaching students to tell how many minutes after the hour, to the nearest five minutes, are shown on the clock. Students need to be proficient in counting by fives before time telling is introduced. When the students can read the minutes after the hour, reading the minutes before the hour can be introduced. Students should be taught to express the time as minutes after and as minutes before the hour (e.g., 40 minutes after 1 is the same as 20 minutes before 2).

- Understanding fractions**.** Creating a fraction to represent the parts of a whole (e.g., $^2/_3$ of a pie) is significantly different from dividing a set of items into subgroups and determining the number of items within some subgroups (e.g., $^2/_3$ of 15). A unit divided into parts can be introduced first, and instruction on that type of fraction should be provided until students can recognize and write fractions to represent fractions of a whole; then the more complex fractions should be introduced. Students can work with diagrams. Computer programs and videos are also available to help with this topic. Students are not expected to solve $^2/_3$ of 15 numerically in the second grade, because doing so requires them to be able to multiply fractions and convert an improper fraction to a whole number.

Chapter 3: Grade Three Areas of Emphasis

By the end of grade three, students deepen their understanding of place value and their understanding of and skill with addition, subtraction, multiplication, and division of whole numbers. Students estimate, measure, and describe objects in space. They use patterns to help solve problems. They represent number relationships and conduct simple probability experiments.

Number Sense

1.0 1.1 1.2 **1.3** 1.4 **1.5**

2.0 **2.1** **2.2** **2.3** **2.4** 2.5 2.6 2.7 2.8

3.0 3.1 **3.2** **3.3** 3.4

Algebra and Functions

1.0 **1.1** 1.2 1.3 1.4 1.5

2.0 **2.1** 2.2

Measurement and Geometry

1.0 1.1 **1.2** **1.3** 1.4

2.0 **2.1** **2.2** **2.3** 2.4 2.5 2.6

Statistics, Data Analysis, and Probability

1.0 1.1 **1.2** **1.3** 1.4

Mathematical Reasoning

1.0 1.1 1.2

2.0 2.1 2.2 2.3 2.4 2.5 2.6

3.0 3.1 3.2 3.3

Chapter 3: Grade Three

Key Standards

NUMBER SENSE

In the Number Sense strand, Standards 1.3 and 1.5 are especially important:

1.3 Identify the place value for each digit in numbers to 10,000.

1.5 Use expanded notation to represent numbers (e.g., 3,206 = 3,000 + 200 + 6).

For students who show a good conceptual understanding of whole numbers (e.g., place value), the second standard should receive special attention. Here, Standards 2.1, 2.2, 2.3, and 2.4 are especially important:

2.1 Find the sum or difference of two whole numbers between 0 and 10,000.

2.2 Memorize to automaticity the multiplication table for numbers between 1 and 10.

2.3 Use the inverse relationship of multiplication and division to compute and check results.

2.4 Solve simple problems involving multiplication of multidigit numbers by one-digit numbers (3,671 x 3 = __).

The foundation that supports standard 2.1 has been laid in grade two: once students become fluent in adding and subtracting three digit numbers, increasing the number of digits offers no real difficulty. The new wrinkle in grade three is standard 2.4. Again, the emphasis at the initial stage of teaching the multiplication algorithm should be on the simple cases where “carrying” plays no role. For example, 234 x 2 is the same as doubling 200 + 30 + 4, which is 400 + 60 + 8, which is 468, which is in turn obtained from 234 by multiplying each digit by 2. The same reasoning applies to 123 x 3. Once students perceive the possibility that the answer to a multi-

digit multiplication might be assembled from the answers to simple single-digit problems, the the idea of "carrying" can be taught, but in assembling the answer to a such a problem as 234 x 6 = 200 x 6 + 30 x 6 + 4 x 6, the fact that the answer can be assembled from the single-digit multiplications 2x6, 3x6, and 4x6 only should be emphasized; it is this fact that makes learning the multiplication table so important.

The relationship between division and multiplication (standard 2.3) should be emphasized from the beginning. In other words, 39 divided by 3 = 13 is the same statement as 39 = 13 x 3. Constant reminder of this fact for children in grade three would seem to be necessary.

Two topics in the third standard also deserve special attention:

3.2 Add and subtract simple fractions (e.g., determine that 1⁄8 + 3⁄8 is the same as 1⁄2).

3.3 Solve problems involving addition, subtraction, multiplication, and division of money amounts in decimal notation and multiply and divide money amounts in decimal notation by using whole-number multipliers and divisors.

These are the early introductory elements of arithmetic with fractions and decimals—topics that will build over several years.

ALGEBRA AND FUNCTIONS

In the third grade, the Algebra and Functions strand grows in importance:

1.1 Represent relationships of quantities in the form of mathematical expressions, equations, or inequalities.

Because understanding these concepts can be a very difficult step for students, instruction must be presented carefully, and many examples should be given: 3 x 12 inches in 3 feet, 4 x 11 legs in 11 cats, 2 x 15 wheels in 15 bicycles, 3 x 15 wheels in

15 tricycles, the number of students in the classroom < 50, the number of days in a year > 300, and so forth.

The next three standards expand on the first and provide examples of what is meant by “represent relationships of . . . .” Teachers must be sure that students are aware of the power of commutativity and associativity in multiplication as a simplifying mechanism and as a means of avoiding overemphasis on pure memorization of the formulas without understanding.

The second standard is also important and likewise must be treated carefully:

2.1 Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

MEASUREMENT AND GEOMETRY

In the first Measurement and Geometry standard, Standards 1.2 and 1.3 should be emphasized:

1.2 Estimate or determine the area and volume of solid figures by covering them with squares or by counting the number of cubes that would fill them.

1.3 Find the perimeter of a polygon with integer sides.

The idea that one cannot talk about area until a square of side 1 has been declared to have unit area and is then used to measure everything else is usually not firmly established in standard textbooks. Analogies should be constantly drawn between length and area. For example, a line segment having a length 3 means that, compared with the segment *L* that has been declared to be of length 1, it can be covered exactly by 3 non-overlapping copies of *L*. Likewise, a rectangle with sides of lengths 3 and 1 has an area equal to 3 because it can be exactly covered by three non-overlapping copies of the square declared to have length 1.

In the second Measurement and Geometry standard, Standards 2.1, 2.2, and 2.3 are the most important.

2.1 Identify, describe, and classify polygons (including pentagons, hexagons, and octagons).

2.2 Identify attributes of triangles
(e.g., two equal sides for the isosceles triangle, three equal sides for the equilateral triangle, right angle for the right triangle).

2.3 Identify attributes of quadrilaterals (e.g., parallel sides for the parallelogram, right angles for the rectangle, equal sides and right angles for the square).

All of these standards can be difficult to master if they are presented too generally. A principal difficulty with geometry at all levels is the need of precise definitions of geometric concepts. Even in grade three, we need a workable definition of a polygon, which textbooks usually do not supply. One may define a polygon as a finite number of line segments, joined end-to-end, so that together they form the complete boundary of a single planar region. It is strongly recommended that the skills for this grade level be limited to such topics as finding the areas of rectangles with integer sides, right triangles with integer sides, and figures that can be partitioned into such rectangles and right triangles. A few examples in which the sides are not whole numbers should also be provided. Estimation should be used for these examples. Implicit in Standards 2.4 and 2.5 is the introduction of the concept of an angle. But this topic should not be emphasized at this time.

STATISTICS, DATA ANALYSIS, AND PROBABILITY

The most important standards for Statistics, Data Analysis, and Probability are:

1.2 Record the possible outcomes for a simple event (e.g., tossing a coin) and systematically keep track of the outcomes when the event is repeated many times.

1.3 Summarize and display the results of probability experiments in a clear and organized way (e.g., use a bar graph or a line plot).

Elaboration

In the third grade, work with addition and subtraction problems expands to problems in which regrouping (i.e., carrying and borrowing) is required in more than one column. As noted earlier particularly important and difficult for some students are subtraction problems that include zeros; for example, 302 - 25 and 3002 - 75 (VanLehn 1990). Students need to become skilled in regrouping across columns with zeros because such problems are often used with money applications; for example, *Jerry bought an ice cream for 62 cents and paid for it with a ten-dollar bill. How much change will he receive?*

One way to treat 302 - 25 is again through the use of the associative law of addition: 302 - 25 = (200 + 102) - 25 = 200 + (102 - 25) = 200 + (2 + 100 - 25) = 200 + (2 + 75) = 277. The first equality is exactly what is meant by “borrowing in the 100s place.”

As with addition and subtraction, memorizing the answers to simple multiplication problems requires the systematic introduction and practice of facts. (Refer to the recommendations discussed for addition facts in the first- grade section.) Some division facts can be incorporated into the sequence for learning multiplication facts. As with addition and subtraction, symmetric relationships can be used to cut down

on the need for memorization. These related facts can be introduced together (20 divided by 5, 5 times 4).

Multiplication and division problems with multidigit terms are introduced in the third grade (e.g., 36 x 5). The basic facts used in both types of problems should have already been committed to memory (e.g., students should have already memorized the answer to 6 x 5, a component of the more complex problem 36 x 5). Students should already be familiar with the basic structure of these problems because of their understanding of how to add a one-digit to a two-digit number (e.g., 18 + 4 and 36 + 5, 12 + 6). As with addition and subtraction, problems that require carrying (e.g., 36 x 5) will be more difficult to solve than will the problems that do not require carrying (e.g., 32 x 4) (Geary 1994).

The goal is to extend the multiplication of whole numbers up to 10,000 by single-digit numbers (e.g., 9,345 x 2) so that students gain mastery of the standard right-to-left multiplication algorithm with the multiplier being a one-digit number.

Students are expected to work on long division problems in which they divide a multidigit number by a single digit. A critical component skill for solving these problems is the ability to determine the multiple of the divisor that is just smaller than the number being divided. In $^{28}/_{5}$, the multiple of 5 that is just smaller than 28 is 25. Although the identification of remainders exceeds the level of the third grade standard, students need to become aware of the process for division when there is a remainder. Practice in determining multiples can be coordinated with the practice of multiplication facts. Having basic multiplication facts memorized will greatly facilitate students' ability to solve these division problems.

Rounding is a critical prerequisite for working estimation problems. Noted in the next column is a sequence of exercises that might be followed when introducing rounding. Each exercise can be introduced over several days, followed by continued

practice. Practice sets should include examples that review earlier stages and present the current ones, as described in Appendix A, “Sample Instructional Profile.”

- Round a 2-digit number to the nearest 10.
- Round a 3-digit number to the nearest 10.
- Round a 3-digit number to the nearest 100.
- Round a 4-digit number to the nearest 1,000.
- Round a 4-digit number to the nearest 100.

The work with fractions in grade three is primarily with diagrams and concrete objects. Students should be able to compare fractions in at least two ways. First, students should be able to order fractions—proper or improper—with like denominators, initially using diagrams but later realizing that if the denominators are equal, then the order depends only on the numerators. Second, students should be able to order unit fractions, perhaps only with whole-number denominators less than or equal to 6. It is not expected at this point that students should compare fractions with unlike denominators except for very simple cases, such as 1⁄4 and 3⁄8 or 1⁄2 and 3⁄4. Students should compare particular fractions verbally and with the symbols <, =, >.

With regard to multiplying and dividing decimals, care should be taken to include exercises in which students have to distinguish between adding and multiplying. Work with money can serve as an introduction to decimals. For example, the following problem is typical of the types of problems that can serve as the introduction of decimal addition:

Josh had $3. He earned $2.50. How much does he have now?

Likewise, the next problem typifies the types of problems that can introduce decimal multiplication:

Josh earned $2.50 an hour. He worked 3 hours. How much did he earn?

The teaching of arithmetic facts can be extended in the third grade to include finding multiples and factors of whole numbers; both are critical to students' understanding of numbers and later to simplifying fractions. Because students need time to develop this skill, it is recommended that they be given considerable instruction on it before they are tested. Only small numbers involving few primes should be used. As a rule, "small" means less than 30, with prime factors limited to only 2, 3, or 5 (e.g., 20 = 2 x 2 x 5, 18 = 3 x 3 x 2).

Considerations for Grade-Level Accomplishments in Grade Three

The most important mathematical skills and concepts for children in grade three to acquire are described as follows:

- Addition and subtraction facts**.** Students who enter the third grade without addition and subtraction facts committed to memory are at risk of having difficulty as more complex mathematics is taught. An assessment of students' knowledge of basic facts needs to be undertaken at the beginning of the school year. Systematic daily practice with addition and subtraction facts needs to be provided for students who have not yet learned them.
- Reading and writing of numbers**.** Thousands numbers with zeros in the hundreds or tens place or both (4006, 4060, 4600) can be particularly troublesome for at-risk students. Systematic presentations focusing on reading and writing thousands numbers with one or two zeros need to be provided until students can read and write these more difficult numbers.
- Rounding off**.** Rounding off a thousands number to the nearest ten, hundred, and thousand requires a sophisticated understanding of the rounding-off process. When rounding to a particular unit, students need to learn at which point to start the rounding process. For example, when rounding off to the nearest hundred,

the student needs to look at the current digit in the tens column to determine whether the digit in the hundreds column will remain the same or be increased when rounded off. Practice items need to include a variety of types (e.g., round off 2,375 to the nearest hundred and then to the nearest thousand).

- Geometry**.** While many of these geometric concepts are not difficult in themselves, students typically have difficulty, becoming confused as new concepts and terms are introduced. This problem is solvable through the use of a cumulative manner of introduction in which previously introduced concepts are reviewed as new concepts are introduced.
- Measurement**.** The standards call for students to learn a significant number of measurement equivalencies. These equivalencies should be introduced so that students are not overwhelmed with too much information at one time. Adequate practice and review are to be provided so that students can readily recall all equivalencies.